

# Applications of a Simple Formula

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## Abstract

New applications of the formula  $A|\psi\rangle = \langle A\rangle|\psi\rangle + \Delta A|\psi_\perp\rangle$  are discussed. Simple derivations of the Heisenberg uncertainty principle and of related inequalities are presented. In addition, the formula is used in an instructive paradox which clarifies a fundamental notion in quantum mechanics.

# 1 Introduction

The topic of this paper concerns a simple formula, rarely mentioned in the literature, which can serve as a helpful tool in quantum mechanics. It has been shown [1] that for any Hermitian operator  $A$  and any quantum state  $|\psi\rangle$ , the following formula is valid:

$$A|\psi\rangle = \langle A \rangle |\psi\rangle + \Delta A |\psi_\perp\rangle, \quad (1)$$

where  $|\psi\rangle$ ,  $|\psi_\perp\rangle$  are normalized vectors,  $\langle \psi_\perp | \psi \rangle = 0$ ,  $\langle A \rangle \equiv \langle \psi | A | \psi \rangle$ , and  $\Delta A \equiv \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ . The proof is as follows: It is always possible to decompose  $A|\psi\rangle = \alpha|\psi\rangle + \beta|\psi_\perp\rangle$  with  $\beta \geq 0$ . Then  $\langle \psi | A | \psi \rangle = \langle \psi | (\alpha|\psi\rangle + \beta|\psi_\perp\rangle)$  yields  $\alpha = \langle A \rangle$ , and  $\langle \psi | A^\dagger A | \psi \rangle = (\alpha^* \langle \psi | + \beta^* \langle \psi_\perp |) (\alpha|\psi\rangle + \beta|\psi_\perp\rangle)$  so that  $\beta = \Delta A$ .

In ref. [1] the formula has been applied to a composite system consisting of a large number of parts in a product state. It was proven that such a product state is essentially an eigenstate of an operator defined as an “average” of variables corresponding to these parts. The formula has also been used in a simple derivation of the minimal time for the evolution of a quantum system to an orthogonal state [2]. Our aim here is to show new applications of this formula. In section 2 an immediate result, related to the uncertainty of an operator in two orthogonal states, is obtained. In section 3 we present an apparent paradox which arises when the formula is used in a naive way. In section 4 we use it to derive in a simple way the Heisenberg uncertainty principle and other related inequalities.

## 2 A Maximal Uncertainty State is Not Unique

Let us rewrite our basic formula in the form

$$A|\psi\rangle = \langle A \rangle_\psi |\psi\rangle + \Delta A_\psi |\psi_\perp\rangle. \quad (2)$$

Then, the scalar product of  $|\psi_\perp\rangle$  and  $A|\psi\rangle$  is

$$\langle \psi_\perp | A | \psi \rangle = \Delta A_\psi. \quad (3)$$

For  $A|\psi_\perp\rangle$  the formula gives

$$A|\psi_\perp\rangle = \langle A \rangle_{\psi_\perp} |\psi_\perp\rangle + \Delta A_{\psi_\perp} |\psi_{\perp\perp}\rangle. \quad (4)$$

where  $\langle \psi_{\perp\perp} | \psi_\perp \rangle = 0$ . Substituting eq.(4) in eq.(3) yields

$$\Delta A_\psi = \Delta A_{\psi_\perp} \langle \psi | \psi_{\perp\perp} \rangle. \quad (5)$$

We see that in the case of a two-dimensional Hilbert space where only two mutually orthogonal states exist,  $\Delta A_\psi = \Delta A_{\psi_\perp}$ . Since  $|\langle \psi | \psi_{\perp\perp} \rangle| \leq 1$ , eq.(5) leads to

$$\Delta A_{\psi_\perp} \geq \Delta A_\psi. \quad (6)$$

Thus, we have proved the following theorem:

For any Hermitian operator  $A$  and any given state  $|\psi\rangle$  there exists a state  $|\psi_\perp\rangle$  orthogonal to  $|\psi\rangle$ , such that  $\Delta A_{\psi_\perp} \geq \Delta A_\psi$ .

This implies that a state corresponding to a maximal uncertainty of any given observable cannot be unique.

### 3 An Apparent Paradox

Let us consider a system described by a two-dimensional Hilbert space, such as a spin- $\frac{1}{2}$  particle. Then, for an Hermitian operator  $B$ , different than  $A$ , a relation similar to eq.(1) holds:

$$B|\psi\rangle = \langle B \rangle |\psi\rangle + \Delta B |\psi_\perp\rangle, \quad (7)$$

where the vector  $|\psi_\perp\rangle$  is the same as in eq.(1) (since it is the only vector orthogonal to  $|\psi\rangle$ ). Note that the quantities  $\langle A \rangle$ ,  $\langle B \rangle$ ,  $\Delta A$  and  $\Delta B$  are all real numbers. Multiplying the Hermitian conjugate of eq.(7) by eq.(1), and using the fact that  $B = B^\dagger$ , we obtain

$$\langle \psi | B A | \psi \rangle = (\langle B \rangle \langle \psi | + \Delta B \langle \psi_\perp |) (\langle A \rangle | \psi \rangle + \Delta A | \psi_\perp \rangle) = \langle B \rangle \langle A \rangle + \Delta B \Delta A. \quad (8)$$

Also, multiplying the Hermitian conjugate of eq.(1) by eq.(7) yields

$$\langle\psi|AB|\psi\rangle = \langle A\rangle\langle B\rangle + \Delta A \Delta B. \quad (9)$$

Thus, subtracting eq.(8) from eq.(9) leads to

$$\langle[A, B]\rangle = 0. \quad (10)$$

Equation (10) states that the expectation value of the commutator of two *arbitrary* operators is always zero, irrespective of the wavefunction which describes the system.

This “remarkable” result can be simply tested using an example of a spin- $\frac{1}{2}$  particle, where  $A = \sigma_x$ ,  $B = \sigma_y$ , and  $|\psi\rangle = |\uparrow_z\rangle$ . The calculation yields

$$\langle[A, B]\rangle = \langle\uparrow_z | [\sigma_x, \sigma_y] | \uparrow_z\rangle = \langle\uparrow_z | 2i\sigma_z | \uparrow_z\rangle = 2i, \quad (11)$$

in contradiction to eq.(10).

A paradox? Not really. Obviously, something is wrong in the derivation of eq.(10). Each of the two basic equations (1) and (7) is separately correct, however, they are not correct when used together. They both use the same vector  $|\psi_\perp\rangle$  orthogonal to  $|\psi\rangle$ , but even in a two-dimensional Hilbert space the orthogonal vector is uniquely defined only up to a phase. If we use the vector  $|\psi_\perp\rangle$  as it is defined by eq.(1), then we should rewrite eq.(7) as follows:

$$B|\psi\rangle = \langle B\rangle|\psi\rangle + \Delta B e^{i\varphi} |\psi_\perp\rangle, \quad (12)$$

where  $\varphi$  is real. Equation (10) is now replaced by the correct expression

$$\langle[A, B]\rangle = \Delta A \Delta B (e^{i\varphi} - e^{-i\varphi}) = 2i \Delta A \Delta B \sin \varphi. \quad (13)$$

Only if  $\varphi = 2\pi n$  where  $n = 0, \pm 1, \pm 2, \dots$ , the expectation value of  $[A, B]$  is equal to zero – not always, as concluded above.

Let us reexamine the previous example with  $A = \sigma_x$ ,  $B = \sigma_y$ , and  $|\psi\rangle = |\uparrow_z\rangle$ .

$$A|\psi\rangle = \sigma_x |\uparrow_z\rangle = |\downarrow_z\rangle, \quad (14)$$

$$B|\psi\rangle = \sigma_y |\uparrow_z\rangle = i|\downarrow_z\rangle. \quad (15)$$

Comparing these equations with eqs.(1) and (12) we obtain  $\Delta\sigma_x = \Delta\sigma_y = 1$  and  $e^{i\varphi} = i$  (that is  $\sin\varphi = 1$ ). Introducing these values in eq.(13) we find

$$\langle \uparrow_z | [\sigma_x, \sigma_y] | \uparrow_z \rangle = 2i, \quad (16)$$

in perfect agreement with eq.(11). Summing up, the phase in quantum mechanics is too important to be neglected.

## 4 The Heisenberg Uncertainty Principle

Our understanding of the preceding apparent paradox has provided us with useful algebraic tools to be implemented in this section. We present here a simple method, based on the formula (eq.(1)), to obtain the Heisenberg uncertainty principle. Consider two Hermitian operators,  $A$  and  $B$ , in an arbitrary Hilbert space. Then, the following equations hold:

$$A|\psi\rangle = \langle A|\psi\rangle + \Delta A|\psi_{\perp A}\rangle, \quad (17)$$

$$B|\psi\rangle = \langle B|\psi\rangle + \Delta B|\psi_{\perp B}\rangle, \quad (18)$$

where  $\langle\psi_{\perp A}|\psi\rangle = 0$  and  $\langle\psi_{\perp B}|\psi\rangle = 0$ . Following the same procedure as in section 3, we find that

$$\langle BA\rangle = \langle B\rangle\langle A\rangle + \Delta B \Delta A \langle\psi_{\perp B}|\psi_{\perp A}\rangle, \quad (19)$$

and

$$\langle AB\rangle = \langle A\rangle\langle B\rangle + \Delta A \Delta B \langle\psi_{\perp A}|\psi_{\perp B}\rangle. \quad (20)$$

Subtracting eq.(19) from eq.(20) yields

$$\begin{aligned} \langle [A, B] \rangle &= \Delta A \Delta B (\langle\psi_{\perp A}|\psi_{\perp B}\rangle - \langle\psi_{\perp B}|\psi_{\perp A}\rangle) \\ &= 2i \Delta A \Delta B \mathcal{Im}\langle\psi_{\perp A}|\psi_{\perp B}\rangle. \end{aligned} \quad (21)$$

Then, taking the absolute value of eq.(21) we find

$$\Delta A \Delta B |\mathcal{Im}\langle\psi_{\perp A}|\psi_{\perp B}\rangle| = \frac{1}{2} |\langle [A, B] \rangle|. \quad (22)$$

The vectors are normalized, therefore  $|\mathcal{I}m\langle\psi_{\perp A}|\psi_{\perp B}\rangle| \leq 1$ , so that we end with

$$\Delta A \Delta B \geq \frac{1}{2} |\langle[A, B]\rangle|, \quad (23)$$

which is the standard form of the uncertainty principle.

Another interesting inequality can be obtained by calculating the anti-commutator of  $A$  and  $B$ . We add eq.(19) to eq.(20) and get

$$\begin{aligned} \langle\{A, B\}\rangle &= 2\langle A\rangle\langle B\rangle + \Delta A \Delta B (\langle\psi_{\perp A}|\psi_{\perp B}\rangle + \langle\psi_{\perp B}|\psi_{\perp A}\rangle) \\ &= 2\langle A\rangle\langle B\rangle + 2\Delta A \Delta B \mathcal{R}e\langle\psi_{\perp A}|\psi_{\perp B}\rangle, \end{aligned} \quad (24)$$

where  $\{A, B\} = AB + BA$ . Rearranging eq.(24) and taking the absolute values of both sides, we find

$$\Delta A \Delta B |\mathcal{R}e\langle\psi_{\perp A}|\psi_{\perp B}\rangle| = \left| \frac{1}{2} \langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle \right|. \quad (25)$$

Since  $|\mathcal{R}e\langle\psi_{\perp A}|\psi_{\perp B}\rangle| \leq 1$ , it follows that

$$\Delta A \Delta B \geq \left| \frac{1}{2} \langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle \right|. \quad (26)$$

This inequality is a by-product of a conventional derivation of the uncertainty principle which is based on the Cauchy-Schwarz inequality [3]. The physical significance of eq.(26) is that it provides an estimate for the correlations developed in time between  $A$  and  $B$ . For instance, it manifests the correlation between  $x$  and  $p$  for the case of a free particle evolving in time [4].

We can also obtain a more accurate expression for  $\Delta A \Delta B$ . Adding eq.(21) to eq.(24) we find

$$\begin{aligned} \langle[A, B]\rangle + \langle\{A, B\}\rangle &= 2i\Delta A \Delta B \mathcal{I}m\langle\psi_{\perp A}|\psi_{\perp B}\rangle \\ &\quad + 2\Delta A \Delta B \mathcal{R}e\langle\psi_{\perp A}|\psi_{\perp B}\rangle + 2\langle A\rangle\langle B\rangle, \end{aligned} \quad (27)$$

and consequently,

$$\Delta A \Delta B \langle\psi_{\perp A}|\psi_{\perp B}\rangle = \frac{1}{2} \langle[A, B]\rangle + \frac{1}{2} \langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle. \quad (28)$$

Taking the norm of both sides we find

$$\Delta A \Delta B \geq \left| \frac{1}{2} \langle [A, B] \rangle + \frac{1}{2} \langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle \right|. \quad (29)$$

Since  $[A, B] = iC$  and  $\{A, B\} = D$ , where  $C$  and  $D$  are Hermitian operators, and since the expectation value of an Hermitian operator is a real number, it follows that

$$\Delta A \Delta B \geq \left[ \left( \frac{1}{2} \langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle \right)^2 + \frac{1}{4} |\langle [A, B] \rangle|^2 \right]^{\frac{1}{2}}. \quad (30)$$

This result combines the two previously found bounds, i.e. eqs.(23) and (26).

## References

- [1] Y. Aharonov and L. Vaidman, “Properties of a Quantum System During the Time Interval Between Two Measurements”, *Phys. Rev. A* **41**, 11 (1990).
- [2] L. Vaidman, “Minimal Time for the Evolution to an Orthogonal State”, *Am. J. Phys.* **60**, 182 (1992).
- [3] See, for instance, S. Wieder, *The Foundations of Quantum Theory* (Academic Press, New York, 1973), pp. 64-65.
- [4] D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, NJ, 1951), pp. 203-207.